Laguerre, Hermite and Meixner polynomials arising from polynomials in several variables

by Bruno Gabutti

Istituto di calcoli numerici – Università degli Studi di Torino, Italy

Communicated by Prof. C.J. Bouwkamp at the meeting of April 26, 1982

SUMMARY

Among polynomials in several variables of given form, the Laguerre and Hermite polynomials are recovered. This is obtained by requiring that the polynomials in several variables satisfy known properties characterizing the classical polynomials.

Analogous characterizations of the Meixner polynomials are likewise determined. These are based on the characteristic properties of the Meixner polynomials recently established.

All new properties generalize the previous ones.

1. INTRODUCTION AND RESULTS

Consider the well-known properties of the Laguerre and Hermite polynomials:

PROPERTY 1. Among the one-variable polynomials the Laguerre polynomials $L_n^{(\alpha)}(x)$ are the only orthogonal polynomials that satisfy a multiplication formula of the form:

$$(1) \qquad \lambda^n f_n(x/\lambda) = \sum_{j=0}^{n} A_{nj}(\lambda-1)^{n-j} f_j(x)$$

with A_{nj} unknown constants ([2]).

PROPERTY 2. The only one-variable polynomials that are both Appell and orthogonal are the Hermite polynomials (see for instance [6]; we recall that $\{p_n(x)\}$ is Appell if and only if $p_n'(x) = n\, p_{n-1}(x)$, $n = 0, 1, \dots$ [5]).

The purpose of this note is to show that these properties, and analogous properties of the Meixner polynomials, ([7] p. 34), can be extended to certain classes of polynomials in n variables. More precisely we will prove the following results, (for $n=2$ see [3]),

THEOREM 1. Among the n-variable polynomials of the form

$$(2) \qquad P_N(x_1,x_2,\dots,x_n)=\sum_{k_1=0}^{N}\sum_{k_2=0}^{N}\cdots\sum_{k_n=0}^{N} p_{N,k_1k_2\dots k_n}x_1^{k_1}x_2^{k_2}\dots x_n^{k_n}$$

the Laguerre polynomials are the only ones satisfying (1) with respect to $x_1,x_2,\dots,x_n$ for any given $x_1,x_2,\dots,x_n$ and orthogonal in x_i, $i\in\{1,2,\dots,n\}$.

THEOREM 2. Among the n-variable polynomials $P_N(x_1,x_2,\dots,x_n)$ of the form (2) the only orthogonal polynomials with respect to x_i, $i\in\{1,2,\dots,N\}$ which also form an Appell set with respect to $x_1,x_2,\dots,x_n$, for all $x_1,x_2,\dots,x_n$, are the Hermite polynomials.

Recently, [3] theorem 1, it has been proved that among the polynomials of two variables $P_N(x_1,x_2)$ ($n=2$ in (2)) the Meixner polynomials are the only ones that enjoy the following properties:

a). $P_N(x_1,x_2)$ satisfy (1) in x_1 for all x_2,

b). $P_N(x_1,x_2)$ are orthogonal in x_2 for all x_1.

An identical result, theorem 2 of [3], holds if *a*) is replaced by *a'*). $P_N(x_1,x_2)$ form an Appell set with respect to x_1 for all x_2.

Actually we can prove the following

THEOREM 3. Among the n-variable polynomials of the form (2) with $n>1$, the Meixner polynomials are uniquely characterized by

A). $P_N(x_1,x_2,\dots,x_n)$ satisfy (1) with respect to $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$ for all $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$.

B). $P_N(x_1,x_2,\dots,x_n)$ are orthogonal with respect to x_i for all $x_1,\dots,x_{i-1}$, $x_{i+1},\dots,x_n$.

The same result holds if instead of *A*) it is required

A'). $P_N(x_1,x_2,\dots,x_n)$ form an Appell set with respect to $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$ for all $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$.

The proofs of the theorems 1, 2, 3 are sketched in the next section. Here we remark that the results of theorems 1, 2, 3 generalize those given in [2], [6], [3], respectively.

For instance if we assume $x_1=x_2=\dots=x_{i-1}=x_{i+1}=\dots=x_n=\text{const.}$ in (2) we see that $P_N(x_1,x_2,\dots,x_n)$ reduces to a polynomial of degree N with respect to x_i and we recover, by theorem 1, Feldheim's result.

An analogous remark applies to theorem 2. Finally, it is immediate to verify that if we put $x_1 = x_2 = \ldots = x_{j-1} = x_{j+1} = \ldots = x_n = \text{const.}$ in (2) and if $j \neq i$ then theorem 3 generalizes the result of [3].

2. PROOF

Before proving any theorems we introduce suitable notations for certain polynomials related to $P_N(x_1, x_2, \ldots, x_n)$ given by (2). This can be written:

$$(3) \quad \begin{cases} P_N(x_1, x_2, \ldots, x_n) = \sum_{k_1=0}^{N} x_1^{k_1} P_{N,k_1}(x_2, x_3, \ldots, x_n) = \\ = \sum_{k_2=0}^{N} x_2^{k_2} P_{N,k_2}(x_1, x_3, \ldots, x_n) = \ldots = \sum_{k_n=0}^{N} x_n^{k_n} P_{Nk_n}(x_1, x_2, \ldots, x_{n-1}). \end{cases}$$

Thus $P_{Nk_i} \equiv P_{Nk_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ are polynomials in $(n-1)$-variables, $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, of degree N (with respect to each variable). In turn P_{Nk_i} can be given explicitly as sum of $(n-1)$-polynomials in $(n-2)$-variables of degree N. For instance:

$$P_{Nk_1}(x_2, x_3, \ldots, x_n) = \sum_{k_i=0}^{N} x_i^{k_i} P_{Nk_1k_i}(x_2, x_3, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \quad i \neq 1$$

$$P_{Nk_2}(x_1, x_3, \ldots, x_n) = \sum_{k_i=0}^{N} x_i^{k_i} P_{Nk_2k_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \quad i \neq 2$$

and so on. More generally we have

$$(4) \quad \begin{cases} P_{Nk_ik_{i+1}\ldots k_n}(x_1, x_2, \ldots, x_i) = \\ = \sum_{k_1=0}^{N} \sum_{k_2=0}^{N} \cdots \sum_{k_i=0}^{N} P_{Nk_1k_2\ldots k_i}(x_{i+1}, x_{i+2}, \ldots, x_n) x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \end{cases}$$

If we conventionally put

$$P_{Nk_1k_2\ldots k_n} = p_{Nk_1k_2\ldots k_n}$$

then equation (2) is recovered by assuming $i = n$ into (4).

All results of this paper are based on the following

LEMMA. If $P_N \equiv P_N(x_1, x_2, \ldots, x_n)$ given by (2) satisfy the multiplication formula (1) with respect to x_i for any λ, $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ then P_N take the form

$$(5) \qquad P_N(x_1, x_2, \ldots, x_n) = Q(x_1, x_2, \ldots, x_n)/C_N$$

with

$$(6) \qquad Q(x_1, x_2, \ldots, x_n) = \sum_{k_1=0}^{N} \sum_{k_2=0}^{N} \cdots \sum_{k_n=0}^{N} q_{k_1k_2\ldots k_n} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$$

and

$$q_{k_1k_2\dots k_n}=p_{Nk_1k_2\dots k_n}/C_N.$$

Here $C_N\neq 0$ is a constant and $p_{Nk_1k_2\dots k_n}$ are the coefficients of P_N (see (2)). Moreover if $Q_{k_i}\equiv Q_{k_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ denotes polynomials of degree k_i, $k_i=0,1,2,\dots,N$, with respect to $x_1,x_2,\dots,x_{i-1},x_{i+1},\dots,x_n$ then the polynomial $Q(x_1,x_2,\dots,x_n)$ can be represented by

$$Q(x_1,x_2,\dots,x_n)=\sum_{k_i=0}^{N} x_i^{k_i}Q_{k_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)/(N-k_i)! \tag{7}$$

PROOF OF THE LEMMA. For any given λ, $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$ it is required that the polynomial $P_N(x_1,\dots,x_i/\lambda,\dots,x_n)$ satisfies

$$\lambda^N P_N(x_1,\dots,x_i/\lambda,\dots,x_n)=\sum_{j=0}^{N} A_{N,N-j}(\lambda-1)^j P_{N-j}(x_1,\dots,x_i,\dots,x_n) \tag{8}$$

where $A_{N,N-j}$ are unknown constants.

By introducing λ into (2) we have, (by using notation (3), (4)),

$$\left\{\begin{aligned} &\lambda^N P_N(x_1,\dots,x_i/\lambda,\dots,x_n)=\\ &=\sum_{k_i=0}^{N}\lambda^{N-k_i}x_i^{k_i}P_{N,k_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)=\\ &=\sum_{k_i=0}^{N}P_{N,k_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)x_i^{k_i}\sum_{j=0}^{N-k_i}\binom{N-k_i}{j}(\lambda-1)^j=\\ &=\sum_{j=0}^{N}(\lambda-1)^j\sum_{k_i=0}^{N-j}\binom{N-k_i}{j}x_i^{k_i}P_{Nk_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\end{aligned}\right. \tag{9}$$

If we equate the coefficients of $(\lambda-1)^j$ of (8), (9) we get

$$A_{N,N-j}P_{N-j}(x_1,x_2,\dots,x_n)=\sum_{k_i=0}^{N}\binom{N-k_i}{j}x_i^{k_i}P_{Nk_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n).$$

By comparing this with

$$P_{N-j}(x_1,x_2,\dots,x_n)=\sum_{k_i=0}^{N-j}x_i^{k_i}P_{N-j,k_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$$

(with respect to x_i) it follows

$$A_{N,m}(N-m)!=\frac{(N-k_i)!P_{N,k_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)}{(m-k_i)!P_{m,k_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)} \tag{10}$$

with

$$k_i=0,1,2,\dots,m;\ m=0,1,2,\dots,N;\ N=0,1,2,\dots$$

These equations, which are to be satisfied for any given $x_1,x_2,\dots,x_{i-1}$, $x_{i+1},\dots,x_n$, imply

$$(11)\qquad (N-k_i)!P_{Nk_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)=C_N Q_{k_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$$

with $Q_{k_i}\equiv Q_{k_i}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ polynomial of degree k_i in $x_1,x_2,\dots,x_{i-1}$, $x_{i+1},\dots,x_n$. This proves (5), (6).

Also, by substituting (11) into (3), and by using (5), we get immediately equation (7); this completes the proof of the lemma.

PROOF OF THEOREM 1. If $P_N(x_1,x_2,\dots,x_n)$ given by (2) satisfies (1) in x_1, the lemma shows that this polynomial has the form (5) with $Q(x_1,x_2,\dots,x_n)$ given by

$$Q(x_1,x_2,\dots,x_n)=\sum_{k_1=0}^{N} x_1^{k_1}Q_{k_1}(x_2,\dots,x_n)/(N-k_1)!$$

where Q_{k_1} is a polynomial of degree k_1 in $x_2,x_3,\dots,x_n$.

Moreover

$$(12)\qquad \begin{cases} Q(x_1,x_2,\dots,x_n)=\displaystyle\sum_{k_1=0}^{N}\frac{x_1^{k_1}}{(N-k_1)!}\sum_{k_2=0}^{k_1} x_2^{k_2}Q_{k_1k_2}(x_3,\dots,x_n)=\\ \qquad =\displaystyle\sum_{k_2=0}^{N} x_2^{k_2}\sum_{k_1=k_2}^{N}\frac{x_1^{k_1}}{(N-k_1)!}Q_{k_1k_2}(x_3,\dots,x_n).\end{cases}$$

In the same way, if it is required that P_N satisfies (1) with respect to x_2, for all $x_1,x_3,\dots,x_n$, then

$$(13)\qquad \begin{cases} Q(x_1,x_2,\dots,x_n)=\displaystyle\sum_{k_2=0}^{N}\frac{x_2^{k_2}}{(N-k_2)!}Q_{k_2}(x_1,x_3,\dots,x_n)=\\ \qquad =\displaystyle\sum_{k_1=0}^{N}\frac{x_2^{k_2}}{(N-k_2)!}\sum_{k_1=0}^{k_2} x^{k_1}Q_{k_2k_1}(x_3,\dots,x_n).\end{cases}$$

The comparaison of (12) with (13) gives

$$Q_{k_2k_1}(x_3,\dots,x_n)=0 \quad \text{if } k_2\neq k_1$$

for all $x_3,\dots,x_n$ and Q_N reduces to

$$Q_N(x_1,x_2,\dots,x_n)=\sum_{k_1=0}^{N}\frac{(x_1x_2)^{k_1}}{(N-k_1)!}Q_{k_1k_2}(x_3,x_4,\dots,x_n).$$

The same argument applied inductively to the remaining variables $x_3,x_4,\dots,x_n$ shows that

$$q_{k_1k_2\dots k_n}=0 \quad \text{if } k_i\neq k_j$$

for some $i, j=1,2,\ldots,n,\ i\neq j$. Thus

$$(14)\qquad Q(x_1,\ldots,x_2,\ldots,x_n)=\sum_{k=0}^{N}\frac{(x_1x_2\ldots x_n)^k}{(N-k)!}\,d_k$$

where

$$(15)\qquad d_k=q_{k_1k_2\ldots k_n}\quad\text{if}\quad k=k_1=k_2=\ldots=k_n.$$

Now, if $Q(x_1,x_2,\ldots,x_n)$ given by (14) is to be orthogonal in x_i then the same problem solved by Feldheim in [2] is recovered and theorem 1 immediately follows. (This procedure is wellknown in orthogonal-polynomials theory; it consists of imposing the polynomial Q to satisfy the three-term recurrence relation and to identify the coefficients of x_i of the same degree. As an example of such a procedure, quite similar to the Feldheim's proof, see proof of theorem 2).

PROOF OF THEOREM 2. We begin by recalling a theorem stated by Carlitz in [1]: "A sequence of polynomials $\{g_n(x)\}$ is an Appell set if and only if the sequence $\{h_n(x)=x^n g_n(x^{-1})\}$ possesses a multiplication formula of the form (1)".

Now, if $P_N(x_1,x_2,\ldots,x_n)$ given by (2) is to be Appell in $x_1,x_2,\ldots,x_n$ for all $x_1,x_2,\ldots,x_n$ then Carlitz's theorem implies that

$$G_N(x_1,x_2,\ldots,x_n)=(x_1x_2\ldots x_n)^N P_N(x_1^{-1},x_2^{-1},\ldots,x_n^{-1})$$

satisfies the multiplication formula (1) with respect to $x_1,x_2,\ldots,x_n$ for all $x_1,x_2,\ldots,x_n$. Thus, the same argument used in the proof of theorem 1 shows that if $G_N(x_1,x_2,\ldots,x_n)$ satisfies the multiplication formula (1) in $x_1,x_2,\ldots,x_n$ for any given $x_1,x_2,\ldots,x_n$ then, apart from an overall multiplicative factor, G_N becomes

$$G_N(x_1,x_2,\ldots,x_n)=\sum_{k=0}^{N}\frac{(x_1x_2\ldots x_n)^k}{(N-k)!}\,d_k$$

where: $d_k=p_{Nk_1k_2\ldots k_n}/C_N$ for $k=k_1=k_2=\ldots=k_n$ (see (5), (15)).

Hence

$$(16)\qquad P_N\equiv P_N(x_1,x_2,\ldots,x_n)=\sum_{k=0}^{N}\frac{(x_1x_2\ldots x_n)^{N-k}}{(N-k)!}\,d_k.$$

It is wellknown (see for instance [4]) that the requirement that P_N is orthogonal with respect to x_i implies that P_N satisfies the recurrence relation

$$x_iP_N=r_NP_{N+1}+s_NP_N+t_NP_{N-1}\qquad N=0,1,2,\ldots$$

where

$$r_N\equiv r_N(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n),\qquad s_N\equiv s_N(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n),$$

$$t_N\equiv t_N(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$

are unknown polynomials in $x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n$.

By equating the coefficients of x_i of the same degree and by using (16) we obtain

$$(x_1x_2...x_n)P_N=(N+1)P_{N+1}+\alpha P_N+\beta P_{N-1}$$

with

$$\alpha=-d_1/d_0,\quad \beta=(d_1^2-2d_0d_2)/d_0^2$$

whence

$$P_N(x_1,x_2,...,x_n)=\frac{1}{N!}\left(\frac{a}{2}\right)^N H_N[(x_1x_2...x_n)/a+b]$$

where

$$a=(2\beta)^{1/2},\quad b=-\alpha/a.$$

In such a way, except for linear changes of variables, the Hermite polynomials are recovered and theorem 2 is proved.

PROOF OF THEOREM 3. Consider firstly hypothesis A) and B). Let $P_N\equiv$ $\equiv P_N(x_1,x_2,...,x_n)$ polynomial of n variables of the form (2); it is required that P_N satisfies the summation formula (1) with respect to $x_1,x_2,...,x_{i-1},x_{i+1},...,x_n$ for any given $x_1,x_2,...,x_{i-1},x_{i+1},...,x_n$.

By using lemma and arguments similar to those considered in the proof of theorem 1 it is easily seen that the resulting polynomials, up to an irrelevant multiplicative factor, take the form

$$P_N(x_1,x_2,...,x_n)=\sum_{k=0}^{N}\frac{(x_1...x_{i-1}x_{i+1}...x_n)^k}{(N-k)!}\,\omega_k(x_i) \tag{17}$$

with $\omega_k(x_i)$ polynomials of degree k in x_i.

The requirement that P_N given by (17) is orthogonal in x_i for all $(x_1x_2,...,x_{i-1},$ $x_{i+1},...,x_n)$ leads to a problem similar to that solved in the proof of theorem 2 (or in [3]); that is, it is imposed that $P_N(x_1,x_2,...,x_n)$ satisfy the three-term recurrence relation with respect to x_i for any $x_1,x_2,...,x_{i-1},x_{i+1},...,x_n$.

It is readily verified that the polynomials P_N defined by (17) are orthogonal in x_i if and only if they belong to the class of the Meixner polynomials.

The proof of theorem 3 with hypothesis A') instead of A) uses arguments of theorem 2 and procedure analogous to the above one; we omit it.

REFERENCES

1. Carlitz, L. – Some multiplication formula, Rendiconti Seminar. Matematico, Università di Padova, **32**, 239-242 (1962).
2. Feldheim, E. – Une proprieté caracteristique des polynomes de Laguerre, Commentarii Mat. Helvetici, **13**, 6-10 (1940).
3. Gabutti, B. – Some characteristic property of the Meixner polynomials, to appear in "Journal Mat. Anal. & Appl".
4. Gatteschi, L. – Funzioni Speciali, UTET, Torino 1973.

5. Scaravelli, C. – Sui polinomi di Appell, Riv. Mat. Univ. Parma, **6**, 103–116 (1965).
6. Shohat, J.A. – The relation of the classical orthogonal polynomials to the polynomials of Appell, Amer. J. Math., **58**, 453–464 (1936).
7. Szegö, G. – Orthogonal Polynomials, Amer. Math. Soc., Colloq. Publ. **23**, Amer. Math. Soc., New York 1967.